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Analytical Solution of the Schrödinger Equation for Makarov Potential with any ℓ Angular Momentum

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Abstract We present the analytical solution of the Schrödinger Equation for the Makarov potential within the framework of the asymptotic iteration method for any n and l quantum numbers. Energy eigenvalues and the corresponding wave functions are calculated. We also obtain the same results for the ring shaped Hartmann potential which is the special form of the non-central Makarov potential.

Keywords Non-central potential · Analytical solution · Ring shaped Hartmann potential

1 Introduction

We consider exactly solvable non-central potential class in the spherical coordinates as follows

$$V(r,\theta) = \frac{\alpha}{r} + \frac{\beta}{r^2 \sin^2\theta} + \frac{\gamma \cos\theta}{r^2 \sin^2\theta}, \tag{1}$$

introduced by Makarov *et al.* [1]. This general potential class reduces to some potential for special case of the potential parameters:

- In case of $\alpha = -Ze^2$, $\beta = 0$ and $\gamma = 0$, Makarov potential reduces to the Coulomb potential which the energy eigenvalues and the eigenfunctions are well known.
- In case of $\alpha = -Ze^2$, $\beta = a - b$ and $\gamma = 0$, Makarov potential reduces to the Aharonov–Bohm (A–B) potential [2]. Here, $a = e^2F^2(\theta)$, $b = 2meF(\theta)$ and $F(\theta) = \frac{\Phi}{2\pi} + g(1 - \cos(\theta))$. Φ and g are the coupling constants.

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- In case of $\alpha = 2a_0\eta\sigma^2\varepsilon_0$, $\beta = -q\eta^2\sigma^2\varepsilon_0 a_0^2$ and $\gamma = 0$, Makarov potential reduces to the ring-shaped Hartmann potential [3].

The non-central Makarov potential have recently been studied by using Nikiforov-Uvarov method [5] both relativistic region in case of equal vector and scalar non-central potential [6] and non-relativistic region [7]. Gönül *et al.* [8] investigate the energy eigenvalues of the Makarov potential and its special case by using supersymmetry and shape invariance [9]. Moreover this potential is also studied by using Path integral method [10, 11] and dynamical group approach [12]. Another type of the noncentral potentials are the oscillatory ring-shaped potential studied by using several techniques [8, 13], and a new noncentral ring-shaped one and anharmonic oscillator ring-shaped one which are studied by using ladder operators with the factorization method [14, 15]. Furthermore, the special case of the Makarov potential is the Aharonov–Bohm (A–B) one which has been studied by using several methods [16–18].

The special case of the non-central Makarov potential is the ring shaped Hartmann potential, which is very important to describe ring-shaped organic molecules like cyclic polyenes and benzene in the quantum chemistry. The ring shaped Hartmann potential, which includes attractive Coulomb and a repulsive inverse square potential, is performed by using several methods both the non-relativistic case [3, 4, 19–22] and the relativistic case [23–25].

In this letter our aim is to obtain the bound state energy eigenvalues and the corresponding eigenfunction of the non-central Makarov potential for any angular momentum quantum number l by using a different and more practical method called the asymptotic iteration method (AIM) [26–28]. In the next section, we present AIM with all necessary formula to perform our calculations. In Sect. 3, we obtain the bound state solution of the Makarov potential. In Sect. 4, we also investigate the energy eigenvalues of the ring shaped Hartmann potential. Finally, Sect. 5 is devoted to the summary and conclusion.

2 The Asymptotic Iteration Method

AIM [26–28] is proposed to solve the second-order differential equations of the form

$$y_n''(x) = \lambda_0(x)y_n'(x) + s_0(x)y_n(x), \tag{2}$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$, are sufficiently differentiable. To find a general solution to this equation, we differentiate (2) with respect to x, we find

$$y_n'''(x) = \lambda_1(x)y_n'(x) + s_1(x)y_n(x), \tag{3}$$

where

$$\begin{aligned} \lambda_1(x) &= \lambda_0'(x) + s_0(x) + \lambda_0^2(x), \\ s_1(x) &= s_0'(x) + s_0(x)\lambda_0(x). \end{aligned} \tag{4}$$

Similarly, the second derivative of the (2) yields

$$y_n^{(4)}(x) = \lambda_2(x)y_n'(x) + s_2(x)y_n(x), \tag{5}$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{6}$$

Equation (2) can be easily iterated up to $(k+1)$th and $(k+2)$th derivatives, $k = 1, 2, 3, \ldots$. Therefore, we have

$$\begin{aligned} y_n^{(k+1)}(x) &= \lambda_{k-1}(x)y_n'(x) + s_{k-1}(x)y_n(x), \\ y_n^{(k+2)}(x) &= \lambda_k(x)y_n'(x) + s_k(x)y_n(x), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x), \end{aligned} \tag{8}$$

which are called as the recurrence relation. From the ratio of the $(k+2)$th and $(k+1)$th derivatives, we have

$$\frac{d}{dx}\ln[y_n^{(k+1)}(x)] = \frac{y_n^{(k+2)}(x)}{y_n^{(k+1)}(x)} = \frac{\lambda_k(x)[y_n'(x) + \frac{s_k(x)}{\lambda_k(x)}y_n(x)]}{\lambda_{k-1}(x)[y_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)}y_n(x)]}. \tag{9}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \tag{10}$$

which is the "asymptotic" aspect of the method, then, (9) is reduced to

$$\frac{d}{dx}\ln[y_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{11}$$

which yields

$$y_n^{(k+1)}(x) = C_1 \exp\left(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)}dx\right) = C_1\lambda_{k-1}(x)\exp\left(\int [\alpha(x) + \lambda_0(x)]dx\right), \tag{12}$$

where C_1 is the integration constant. By inserting (12) into (7), the first-order differential equation is obtained as

$$y_n'(x) + \alpha(x)y_n(x) = C_1 \exp\left(\int [\alpha(x) + \lambda_0(x)]dx\right). \tag{13}$$

This first order differential equation can easily be solved and the general solution of the (2) can be obtained as:

$$y_n(x) = \exp\left(-\int^x \alpha(x_1)dx_1\right)\left[C_2 + C_1\int^x \exp\left(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)]dx_2\right)dx_1\right]. \tag{14}$$

For a given potential, the Schrödinger equation is converted to the form of (2). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (8). The termination condition of the method in (10) can be arranged as

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \quad k = 1, 2, 3, \ldots, \tag{15}$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of the (15) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential [26–31] and the approximate energy eigenvalues are obtained from the roots of the (15) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of (2) is given by (14). The first part of (14) gives us the polynomial solutions that are convergent and physical, whereas the second part of (14) gives us non-physical solutions that are divergent. Although (14) is the general solution of (2), we take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials:

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)} dx_1\right), \tag{16}$$

where n represents the radial quantum number.

3 Analytical Solution

The motion of a particle with the mass μ in the spherically symmetric potential is described in the spherical coordinates by the following Schrödinger equation:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{2\mu}{\hbar^2}\left(E - V(r,\theta,\varphi)\right)\right]\psi(r,\theta,\varphi) = 0. \tag{17}$$

If one assigns the corresponding spherical total wave function as $\psi(r,\theta,\varphi)=\frac{1}{r}R(r)Y(\theta,\varphi)$, then by selecting $Y(\theta,\varphi)=\frac{1}{\sin^{1/2}\theta}H(\theta)\Phi(\varphi)$, the wave (17) for a general non-central potential is separated into variables and the following equations are obtained:

$$\frac{d^2R(r)}{dr^2} + \left(\frac{2\mu}{\hbar^2}\left(E - V(r)\right) - \frac{l(l+1)}{r^2}\right)R(r) = 0, \tag{18}$$

$$\frac{d^2H(\theta)}{d\theta^2} - \left(\frac{2\mu}{\hbar^2}V(\theta) + \frac{m^2}{\sin^2\theta} - \frac{1}{2} - \frac{1}{4}\frac{\cos^2\theta}{\sin^2\theta} - l(l+1)\right)H(\theta) = 0, \tag{19}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2\Phi(\varphi), \tag{20}$$

where $l = 0, 1, 2, \ldots, n-1$ and $m = 0, \pm 1, \pm 2, \ldots, \pm l$ are orbital and azimuthal quantum numbers, respectively. For bound states, we have the boundary conditions $R(0) = 0$ and $R(\infty) = 0$ in (18), $H(0)$ and $H(\pi)$ are infinite in (19) and $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ in (20). If we specialize to the case where $V(\varphi) = 0$, the normalized solution of the (20) that provides the boundary condition is

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}. \tag{21}$$

We consider solutions of radial and angle-dependent parts of Schrödinger equation for the non-central Coulomb potential within the framework of the asymptotic iteration method. The radial Schrödinger equation for the non-central Coulomb potential can be written as,

$$\frac{d^2R(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left(E - \frac{\alpha}{r} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right) R(r) = 0. \tag{22}$$

If the following abbreviations are used:

$$-\varepsilon^2 = \frac{2\mu E}{\hbar^2}, \qquad \tilde{\alpha} = -\frac{2\mu\alpha}{\hbar^2}. \tag{23}$$

The radial Schrödinger equation takes the following form which is convenient in order to apply AIM.

$$\frac{d^2R(r)}{dr^2} + \left(-\varepsilon^2 + \frac{\tilde{\alpha}}{r} - \frac{l(l+1)}{r^2}\right) R(r) = 0. \tag{24}$$

In order to solve this equation with AIM for $l \neq 0$, we should transform this equation to the form of (2). Therefore, the reasonable physical wave function we propose is as follows

$$R(r) = r^{l+1} e^{-\varepsilon r} f(r). \tag{25}$$

If we insert this wave function into the (24), we have the second-order homogeneous linear differential equations in the following form

$$\frac{d^2f(r)}{dr^2} = \left(2\varepsilon - \frac{2(l+1)}{r}\right)\frac{df(r)}{dr} + \left(\frac{2\varepsilon(l+1) - \tilde{\alpha}}{r}\right) f(r), \tag{26}$$

which is now amenable to an AIM solution. To apply the AIM, it is required to compare (26) with (2). Subsequently, by using (8), the values of $\lambda_n(r)$ and $s_n(r)$ are computed as follows

$$\begin{aligned}
\lambda_0(r) &= 2\varepsilon - \frac{2(l+1)}{r}, \\
s_0(r) &= \frac{2\varepsilon(l+1) - \tilde{\alpha}}{r}, \\
\lambda_1(r) &= \frac{2l(2l+5)+6}{r^2} - \frac{6\varepsilon(l+1) + \tilde{\alpha}}{r} + 4\varepsilon^2, \\
s_1(r) &= \frac{3\tilde{\alpha} + 2\tilde{\alpha}l - 4\varepsilon l^2 - 6\varepsilon - 10\varepsilon l}{r^2} + \frac{4\varepsilon^2(l+1) - 2\tilde{\alpha}\varepsilon}{r}, \\
&\ldots \text{etc.}
\end{aligned} \tag{27}$$

If we use the termination condition of the AIM given in (15), energy eigenvalues are obtained as follows

$$\begin{aligned} s_0\lambda_1 - s_1\lambda_0 = 0 \quad &\Rightarrow \quad \varepsilon_0 = \frac{\tilde{\alpha}}{2(l+1)}, \\ s_1\lambda_2 - s_2\lambda_1 = 0 \quad &\Rightarrow \quad \varepsilon_1 = \frac{\tilde{\alpha}}{2(l+2)}, \\ s_2\lambda_3 - s_3\lambda_2 = 0 \quad &\Rightarrow \quad \varepsilon_2 = \frac{\tilde{\alpha}}{2(l+3)}, \\ &\ldots \text{etc.}, \end{aligned} \tag{28}$$

which can be generalized as

$$\varepsilon_{nl} = \frac{\tilde{\alpha}}{2(n+l+1)}, \quad n = 0, 1, 2 \ldots. \tag{29}$$

Using (23) and (29), we obtain the energy eigenvalues of the Coulomb potential E_{nl},

$$E_{nl} = -\frac{\mu}{2\hbar^2}\frac{\alpha^2}{(n+l+1)^2}, \tag{30}$$

where n and l are radial quantum number and angular momentum quantum number and $\alpha = -Ze^2$. Here, Z and e are the atomic number and electron charge, respectively.

In order to find the corresponding energy eigenfunctions, we may use (16). Below, the first few $f_{nl}(r)$ functions can be seen

$$\begin{aligned} &f_{0l}(r) = 1, \\ &f_{1l}(r) = (l+2)(2l+2)\left(1 - \frac{2\varepsilon_{1l}r}{(2l+2)}\right), \\ &f_{2l}(r) = (2l+2)(2l+3)(l+3)^2\left(1 - \frac{4\varepsilon_{2l}r}{(2l+2)} + \frac{4\varepsilon_{2l}^2 r^2}{(2l+2)(2l+3)}\right), \\ &\ldots \text{etc.} \end{aligned} \tag{31}$$

It may be understood from the results given above that we can write the general formula for $f_{nl}(r)$ as follows

$$f_{nl}(r) = (l+n+1)^n\left(\prod_{k=0}^{n-1}(2l+2+k)\right)\ {}_1F_1(-n, 2l+2; 2\varepsilon_{nl}r). \tag{32}$$

Therefore, we can write the total radial wave function as below:

$$R_{nl}(r) = C_2 r^{l+1} e^{-\varepsilon_{nl}r}\ {}_1F_1(-n, 2l+2; 2\varepsilon_{nl}r), \tag{33}$$

where C_2 is the normalization constant.

We now investigate solutions of the angle-dependent Schrödinger equation with the non-central Coulomb potential using the same approach. Inserting angular part of (1) into (19) and defining $l = \Lambda - \frac{1}{2}$ and $l(l+1) = \Lambda^2 - \frac{1}{4}$, (19) can be written as

$$\frac{d^2H(\theta)}{d\theta^2} + \left[\Lambda^2 - \frac{\frac{2\mu\beta}{\hbar^2} + m^2 - \frac{1}{4}}{\sin^2\theta} - \frac{2\mu\gamma}{\hbar^2}\frac{\cos\theta}{\sin^2\theta}\right]H(\theta) = 0. \tag{34}$$

This equation may be further arranged as follows

$$\frac{d^2H(\theta)}{d\theta^2}+\left[\Lambda^2-\frac{\kappa^2+\eta^2-\frac{1}{4}}{\sin^2\theta}-\frac{2\kappa\eta\cos\theta}{\sin^2\theta}\right]H(\theta)=0 \tag{35}$$

with

$$\kappa^2=\frac{1}{2}\left[\frac{2\mu\beta}{\hbar^2}+m^2+\sqrt{\left(m^2+\frac{2\mu\beta}{\hbar^2}\right)^2-\left(\frac{2\mu\gamma}{\hbar^2}\right)^2}\right], \tag{36}$$

$$\eta^2=\frac{1}{2}\left[\frac{2\mu\beta}{\hbar^2}+m^2-\sqrt{\left(m^2+\frac{2\mu\beta}{\hbar^2}\right)^2-\left(\frac{2\mu\gamma}{\hbar^2}\right)^2}\right]. \tag{37}$$

Equation (35) can be written in the following form by introducing a new variable of the form $x=\cos\theta$,

$$\frac{d^2H(x)}{dx^2}-\frac{x}{1-x^2}\frac{dH(x)}{dx}+\left[\frac{\Lambda^2}{(1-x^2)}-\frac{\kappa^2+\eta^2-\frac{1}{4}}{(1-x^2)^2}-\frac{2\kappa\eta x}{(1-x^2)^2}\right]H(x)=0. \tag{38}$$

Let the angular wave function be factorized as:

$$H(x)=(1-x)^{(\frac{\kappa+\eta}{2}+\frac{1}{4})}(1+x)^{(\frac{\kappa-\eta}{2}+\frac{1}{4})}f(x). \tag{39}$$

Inserting this wave function into the (38), we have the second-order homogeneous linear differential equations in the following form

$$\frac{d^2f(x)}{dx^2}=2\left(\frac{\eta+(\kappa+1)x}{1-x^2}\right)\frac{df(x)}{dx}+\left(\frac{(\kappa+1/2)^2-\Lambda^2}{1-x^2}\right)f(x), \tag{40}$$

which is now amenable to an AIM solution. By comparing this equation with (2), we can write the $\lambda_0(x)$ and $s_0(x)$ values and by means of (8), we may calculate $\lambda_k(x)$ and $s_k(x)$. This gives:

$$\begin{aligned}
\lambda_0(x)&=\frac{2\eta+(2\kappa+2)\,x}{1-x^2},\\
s_0(x)&=\frac{(\kappa+1/2)^2-\Lambda^2}{1-x^2},\\
\lambda_1(x)&=\frac{(23+36\kappa+4\Lambda^2+12\kappa^2)x^2+(32\kappa+48)\eta x}{4(x^2-1)^2}\\
&\quad+\frac{9+12\kappa-4\Lambda^2+4\kappa^2+16\eta^2}{4(x^2-1)^2},\\
s_1(x)&=\frac{(1-4\Lambda^2+4\kappa+4\kappa^2)(2x+\kappa x+\eta)}{2(x^2-1)^2},\\
&\ldots\text{etc.}
\end{aligned} \tag{41}$$

Combining these results with the condition given by (15) yields

$$\begin{aligned} s_0\lambda_1 - s_1\lambda_0 = 0 \quad &\Rightarrow \quad \Lambda^2 = \left(\kappa + \frac{1}{2}\right)^2, \\ s_1\lambda_2 - s_2\lambda_1 = 0 \quad &\Rightarrow \quad \Lambda^2 = \left(\kappa + \frac{3}{2}\right)^2, \\ s_2\lambda_3 - s_3\lambda_2 = 0 \quad &\Rightarrow \quad \Lambda^2 = \left(\kappa + \frac{5}{2}\right)^2, \\ &\ldots \text{etc.} \end{aligned} \tag{42}$$

When the above expressions are generalized, the eigenvalues turn out as

$$\Lambda^2 = \left(\kappa + N + \frac{1}{2}\right)^2, \quad N = 0, 1, 2, \ldots. \tag{43}$$

By inserting (36) and Λ into (43)

$$l = \sqrt{\frac{\frac{2\mu\beta}{\hbar^2} + m^2 + \sqrt{(m^2 + \frac{2\mu\beta}{\hbar^2})^2 - (\frac{2\mu\gamma}{\hbar^2})^2}}{2}} + N. \tag{44}$$

After inserting (44) into (30), we can obtain the energy eigenvalues of the noncentral Coulomb potential as follows for any n, l and m quantum numbers

$$E_{nlm} = -\frac{\mu}{2\hbar^2} \frac{\alpha^2}{\left(n + N + 1 + \sqrt{\frac{\frac{2\mu\beta}{\hbar^2} + m^2 + \sqrt{(m^2 + \frac{2\mu\beta}{\hbar^2})^2 - (\frac{2\mu\gamma}{\hbar^2})^2}}{2}}\right)^2}. \tag{45}$$

Now, we can find the $f_N(x)$ by using the wave function generator, (16). Below, the first few $f_N(x)$ functions can be seen

$$\begin{aligned} f_0(x) &= \frac{C_2}{\sqrt{\pi}}, \\ f_1(x) &= -C_2\left(\frac{1}{\sqrt{\pi}} - \left(\frac{3}{\sqrt{\pi}} + \frac{2\kappa}{\sqrt{\pi}}\right)x^2\right), \\ f_2(x) &= 3C_2\left(\frac{1}{\sqrt{\pi}} - \left(\frac{10}{\sqrt{\pi}} + \frac{4\kappa}{\sqrt{\pi}}\right)x^2 + \left(\frac{35}{3\sqrt{\pi}} + \frac{8\kappa}{\sqrt{\pi}} + \frac{4\kappa^2}{3\sqrt{\pi}}\right)x^4\right), \\ &\ldots \text{etc.} \end{aligned} \tag{46}$$

It may be understood from the results given above that we can write the general formula for $f_n(x)$ as follows

$$f_N(x) = (-1)^N C_2 2^N \frac{\Gamma(N + \frac{1}{2})}{\Gamma(\frac{1}{2})} {}_2F_1\left(-N, \kappa + \frac{1}{2} + N; \frac{1}{2}; x^2\right). \tag{47}$$

Therefore, we can write $H(x)$ part of the total wave function as below,

$$H(x) = C_3(1-x)^{(\frac{\kappa+\eta}{2}+\frac{1}{4})}(1+x)^{(\frac{\kappa-\eta}{2}+\frac{1}{4})} {}_2F_1\left(-N, \kappa + \frac{1}{2} + N; \frac{1}{2}; x^2\right), \tag{48}$$

where C_3 is the normalization constant.

Therefore, we may rewrite the total wave function $\psi(r,\theta,\varphi)=\frac{1}{r}R(r)Y(\theta,\varphi)$ by using (21), (33) and (48) as follows

$$\begin{aligned}\psi_{nlm}(r,\theta,\varphi) = {} & Cr^l e^{-\varepsilon_{nl}r}\,{}_1F_1(-n,2l+2;2\varepsilon_{nl}r)\\ & \times(1-\cos(\theta))^{(\frac{\kappa+\eta}{2}+\frac{1}{4})}(1+\cos(\theta))^{(\frac{\kappa-\eta}{2}+\frac{1}{4})}\\ & \times {}_2F_1\left(-N,\kappa+\frac{1}{2}+N;\frac{1}{2};\cos(\theta)^2\right)\times e^{im\varphi},\end{aligned} \tag{49}$$

where C is the normalization constant.

4 Hartmann Ring-Shaped Potential

In the (1), selecting $\alpha = 2a_0\eta\sigma^2\varepsilon_0$, $\beta = -q\eta^2\sigma^2\varepsilon_0 a_0^2$ and $\gamma = 0$, we obtain

$$V(r,\theta) = \eta\sigma^2\varepsilon_0\left(\frac{2a_0}{r} - q\eta\frac{a_0^2}{r^2\sin^2\theta}\right), \tag{50}$$

which called the ring-shaped Hartmann potential that was introduced in quantum chemistry to describe ring-shaped organic molecules like benzene. Here, $a_0 = \frac{\hbar^2}{\mu e^2}$ is Bohr radius and $\varepsilon_0 = -\frac{\mu e^4}{2\hbar^2}$ is the ground state energy of the Hydrogen atom. η, σ and q are tree dimensionless parameters. Therefore, α and β parameters become $\alpha = -\eta\sigma^2 e^2$ and $\beta = q\eta^2\sigma^2\frac{\hbar^2}{2\mu}$. Inserting α, β and γ values into (45), we can immediately obtain the energy eigenvalues of the ring shaped Hartmann potential as follows

$$E_n = -\frac{\mu}{2\hbar^2}\frac{(\eta\sigma^2e^2)^2}{(n+N+1+\sqrt{m^2+q\eta^2\sigma^2})^2}. \tag{51}$$

5 Conclusion

In this paper we have investigated the energy eigenvalues and the corresponding eigenfunctions of the non-central Makarov potential by using asymptotic iteration method for any n and l quantum numbers. The Makarov potential is a general potential class and reduced to the ring shaped Hartmann potential. Therefore, we have also investigated the analytical solution of the ring shaped Hartmann potential and obtained the energy eigenvalues in the closed-form.

The method presented in this study is a systematic one and it is very efficient and practical. It is worth extending this method to the solution of other non-central potential class.

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